Entropy stable modal discontinuous Galerkin schemes and wall boundary conditions for the compressible Navier-Stokes equations

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Abstract

In this work, we describe a discretization of viscous terms in the compressible Navier-Stokes equations which enables a simple and explicit imposition of entropy stable boundary conditions for discontinuous Galerkin (DG) discretizations.

Background

In this work, we focus on the compressible Navier-Stokes equations in 2D:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \boldsymbol{f}_i}{\partial x_i} = \sum_{i=1}^{d} \frac{\partial \boldsymbol{g}_i}{\partial x_i}, \tag{1}$$

whose entropy variables $\boldsymbol{v}(\boldsymbol{u})$ symmetrizes the viscous fluxes [2] in the sense that

$$\sum_{i=1}^{d} \frac{\partial \boldsymbol{g}_i}{\partial x_i} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{v}}{\partial x_j}\right). \tag{2}$$

where $\boldsymbol{K}_{ij}$ denote blocks of a symmetric and positive semi-definite matrix $\boldsymbol{K}$. The continuous entropy balance for the compressible Navier-Stokes equation is

$$\int_{\Omega} \frac{\partial S(\boldsymbol{u})}{\partial t} = \int_{\partial\Omega} \sum_{i=1}^{d} \left(\frac{1}{c_v T}\kappa \frac{\partial T}{\partial x_i} - F_i(\boldsymbol{u})\right) n_i \tag{3}$$

$$- \int_{\Omega} \sum_{i,j=1}^{d} \left(\frac{\partial \boldsymbol{v}}{\partial x_i}\right)^T \left(\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{v}}{\partial x_j}\right). \tag{4}$$

The goal of our work will be to impose boundary conditions such that the semi-discrete entropy inequality mimics the continuous entropy balance.

Discretization of inviscid terms

The inviscid terms are discretized using a "flux differencing" approach involving summation-by-parts (SBP) operators and entropy conservative fluxes. Extending from nodal to modal formulation requires the hybridized SBP operator [1]

$$\mathbf{Q}_{i,h} = \frac{1}{2}\begin{bmatrix} \mathbf{Q}_i - (\mathbf{Q}_i)^T & \mathbf{E}^T\mathbf{B}_i \\ \mathbf{B}_i\mathbf{E} & \mathbf{B}_i \end{bmatrix}. \tag{5}$$

and "entropy projected" conservative variables

$$\mathbf{v} = \mathbf{P}_q \boldsymbol{v}(\mathbf{V}_q \mathbf{u}), \qquad \widetilde{\mathbf{u}} = \boldsymbol{u}(\mathbf{V}_h \mathbf{v}) \tag{6}$$

Then, the inviscid term is discretized by

$$\frac{\partial \boldsymbol{f}_i(\boldsymbol{u})}{\partial x_i} \Longleftrightarrow \mathbf{V}_h^T\left(2\mathbf{Q}_{i,h}^k \circ \mathbf{F}_i\right)\mathbf{1}, \qquad (\mathbf{F}_i)_{jk} = \boldsymbol{f}_{i,S}(\widetilde{\boldsymbol{u}}_i, \widetilde{\boldsymbol{u}}_j). \tag{7}$$

Discretization of viscous terms, imposition of boundary conditions

We discretize the symmetrized viscous terms using a local DG formulation:

$$\boldsymbol{\Theta}_i \approx \frac{\partial \boldsymbol{v}}{\partial x_i}, \qquad (\boldsymbol{\Theta}_i, \boldsymbol{w}_{1,i})_{D^k} = \left(\frac{\partial \boldsymbol{v}}{\partial x_i}, \boldsymbol{w}_{1,i}\right)_{D^k} + \frac{1}{2}\langle [\![\boldsymbol{v}]\!] n_i, \boldsymbol{w}_{1,i}\rangle_{\partial D^k} \tag{8}$$

$$\boldsymbol{\sigma}_i \approx \sum_{j=1}^{d} \boldsymbol{K}_{ij}\frac{\partial \boldsymbol{v}}{\partial x_j}, \qquad (\boldsymbol{\sigma}_i, \boldsymbol{w}_{2,i})_{D^k} = \left(\sum_{j=1}^{d} \boldsymbol{K}_{ij}\boldsymbol{\Theta}_j, \boldsymbol{w}_{2,i}\right)_{D^k} \tag{9}$$

$$\boldsymbol{g}_{\rm visc} \approx \sum_{i=1}^{d} \frac{\partial \boldsymbol{g}_i}{\partial x_i}, \qquad (\boldsymbol{g}_{\rm visc}, \boldsymbol{w}_3)_{D^k} = \sum_{i=1}^{d}\left[\left(-\boldsymbol{\sigma}_i, \frac{\partial \boldsymbol{w}_3}{\partial x_i}\right)_{D^k} + \langle \{\{\boldsymbol{\sigma}_i\}\} n_i, \boldsymbol{w}_3\rangle_{\partial D^k}\right] - \langle \boldsymbol{\tau}_{\rm visc}[\![\boldsymbol{v}]\!], \boldsymbol{w}_3\rangle_{\partial D^k}, \tag{10}$$

For periodic boundary conditions, the viscous entropy satisfy $(\boldsymbol{g}_{\rm visc}, \boldsymbol{v}) \leq 0$.
This formulation enables a simple and explicit imposition of entropy stable boundary conditions:

Adiabatic no-slip wall BC

$$\begin{aligned} v_{1+i}^+ &= -2u_{i,\rm wall}v_4 - v_{1+i} \\ v_4^+ &= v_4 \\ \sigma_{2,i}^+ &= \sigma_{2,i}, \\ \sigma_{3,i}^+ &= \sigma_{3,i}, \\ \sigma_{4,i}^+ &= 2(u_{1,\rm wall}\sigma_{2,i} + u_{2,\rm wall}\sigma_{3,i} \\ &\quad + \frac{c_v g(t) n_i}{v_4}) - \sigma_{4,i} \end{aligned}$$

Isothermal no-slip wall BC

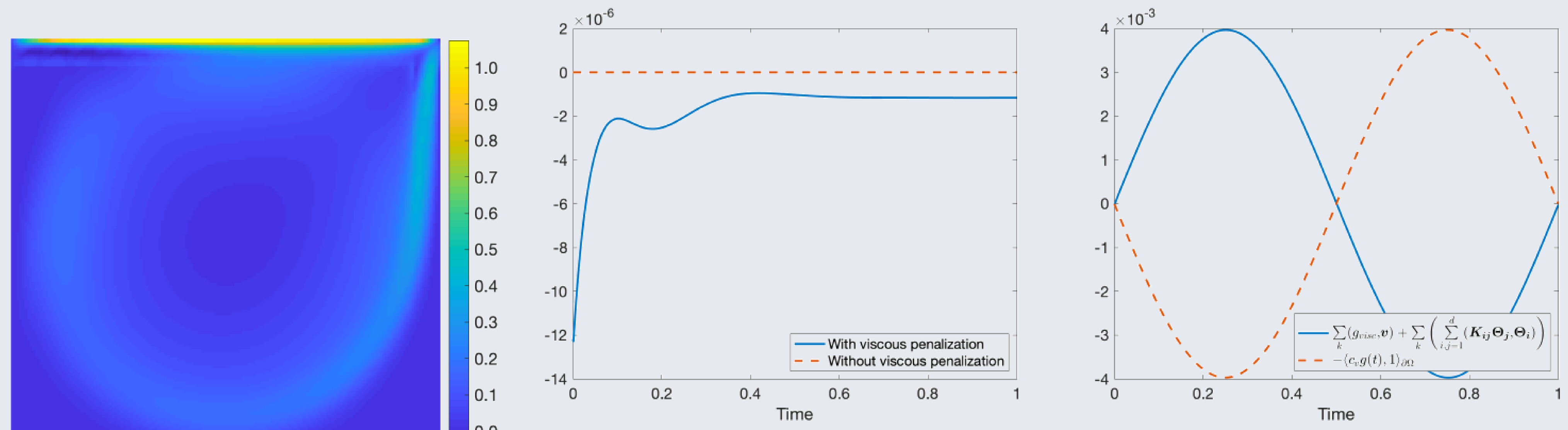
$$\begin{aligned} v_{1+i}^+ &= \frac{2u_{i,\rm wall}}{c_v T_{\rm wall}} - v_{1+i} \\ v_4^+ &= -\frac{2}{c_v T_{\rm wall}} - v_4 \\ \sigma_{2,i}^+ &= \sigma_{2,i}, \\ \sigma_{3,i}^+ &= \sigma_{3,i}, \\ \sigma_{4,i}^+ &= \sigma_{4,i} \end{aligned}$$

Reflective wall BC

$$\begin{aligned} v_{1+i}^+ &= v_{1+i} - 2v_n n_i \\ v_4^+ &= v_4 \\ \sigma_{1+i,j}^+ &= 2n_i\sigma_{n,j} - \sigma_{1+i,j} \\ \sigma_{4,i}^+ &= -\sigma_{4,i} \end{aligned}$$

The proposed boundary conditions satisfy the corresponding continuous entropy balance.

Numerical experiments



(a) $r(t)$ over time for $g(t) = 0$ (b) $r(t)$ for $g(t) = 10^{-4}\sin(4\pi x)$

Figure: Norm of velocity and evolution of $r(t)$ for the lid-driven cavity problem under zero (adiabatic) and non-zero heat entropy flow

We solve the lid-driven cavity problem with $\mathrm{Ma} = .1, \mathrm{Re} = 1000, N = 3$, and $K_{\rm 1D} = 16$ to compute the "viscous entropy residual" $r(t)$

$$r(t) = \sum_k \left[(\boldsymbol{g}_{\rm visc}, \boldsymbol{v})_{D^k} + \sum_{i,j=1}^{d} (\boldsymbol{K}_{ij}\boldsymbol{\Theta}_j, \boldsymbol{\Theta}_i)\right] \tag{11}$$

Numerical experiments

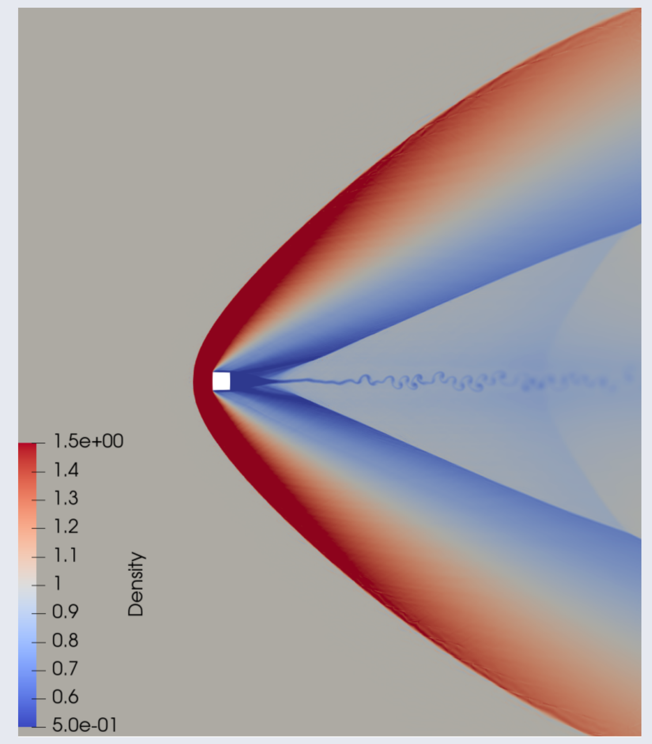


Figure: Supersonic flow over a square cylinder

We investigate the supersonic flow from a square cylinder. We take $\mathrm{Re} = 10^4$ and $\mathrm{Ma} = 1.5$ and impose zero adiabatic no-slip solid wall boundary conditions on the cylinder wall. Shocks and trailing vortices behind the square cylinder are both visible in the numerical simulation, and the simulation remains stable without additional artificial viscosity or limiting.

Conclusion

In this work, we

- present an entropy stable approach for discretizing viscous term.
- develop entropy stable imposition of various wall boundary conditions under the proposed framework.

References

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